

MEM6810 Engineering Systems Modeling and Simulation



工程系统建模与仿真

Theory Analysis

Lecture 4: Random Variate Generation

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Sino-US Global Logistics Institute (Institute of Industrial & System Engineering)



- 1 Introduction
- 2 Random Number Generation
 - ▶ Definition
 - ▶ Pseudo-Random Numbers
 - ▶ Linear Congruential Generator
 - ▶ More Sophisticated RNGs
 - ▶ Tests for Random Numbers
- 3 Random Variate Generation
 - ▶ Inverse-Transform Technique
 - ▶ Acceptance-Rejection Technique
 - ▶ Other Ad-Hoc Methods
 - ▶ Generating Poisson Process



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 - E.g., 5 random variates (outcomes) from a $\mathcal{N}(0, 1)$ random variable: 0.5377, 1.8339, -2.2588, 0.8622, 0.3188.
- When simulating a system, we often need to generate random variates (e.g., interarrival time, service time) from all kinds of distributions (e.g., exponential distribution, arbitrary empirical distribution).

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 - Most simulation softwares have build-in functions to generate random variates from common distributions.
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 - To better understand the randomness in stochastic simulation.
 - Be alert to some inadequate random variate generator.
- To produce a sequence of random variates from a given distribution (of a random variable):
 - ① Start with random variates from $\text{Unif}(0, 1)$ (called **random numbers**).
 - ② All random variates with given distribution are “transformed” from **random numbers**.

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 - If $U \sim \text{Unif}(0, 1)$, then $\mathbb{E}[U] = \frac{1}{2}$, $\text{Var}(U) = \frac{1}{12}$, and its pdf is

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- Statistical Properties
 - Uniformity: Each value on $[0, 1]$ has equal likelihood.
 - Independence: Implies no correlation between successive numbers.

- Uniformity

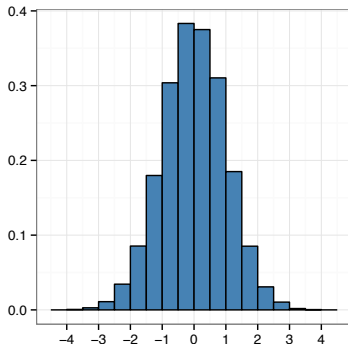
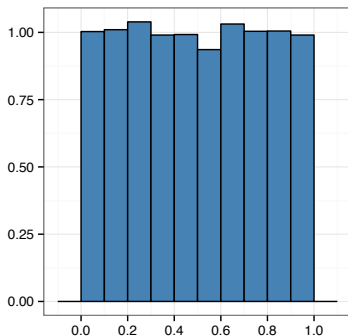


Figure: Empirical pdf (i.e., Scaled Histogram): Uniformity vs Nonuniformity (from [ZHANG Xiaowei](#))

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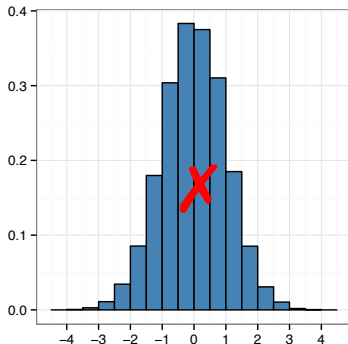
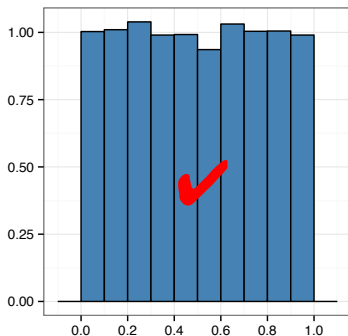


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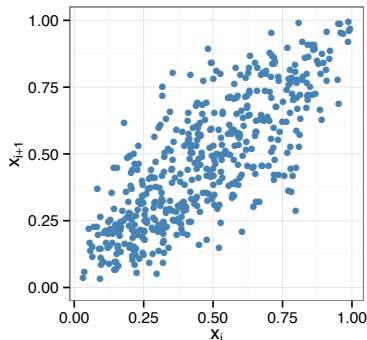
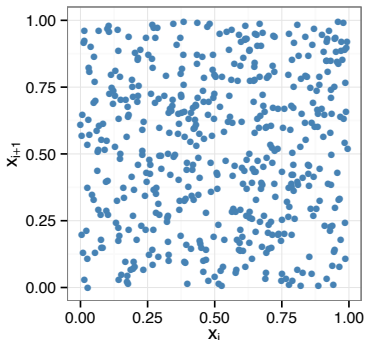


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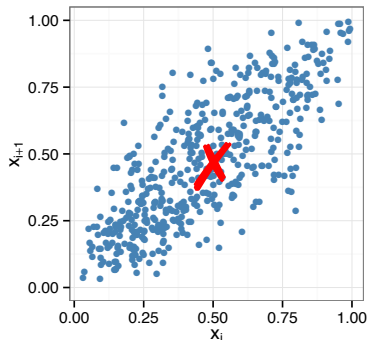
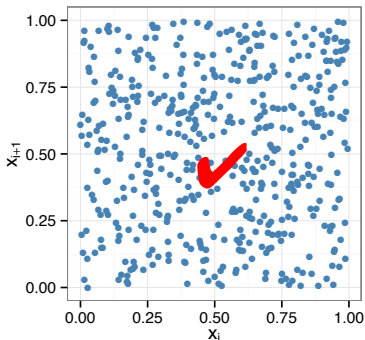


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 - Generating random numbers by a known method removes true randomness.
 - The set of pseudo-random numbers can be repeated.
- Goal: To produce a sequence of numbers in $[0, 1]$ that imitates the ideal properties of random numbers.
 - Statistical properties are the most important.
 - True randomness is not the first priority.

- Properties of a good random number generator (RNG):
 - ① Pass statistical tests.
 - ② Solid theoretical support.
 - ③ Fast.
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 - ⑤ Portable to different computers.
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 - Linear Congruential Generator (LCG)
 - Combined LCG
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- Possible values of u_i : $\{0, \frac{1}{m}, \dots, \frac{m-1}{m}\}$. (May not cover all!)
- The selection of the values for a , c , m , and x_0 drastically affects the statistical properties and the cycle length.

- Example: Use LCG with $x_0 = 27$, $a = 17$, $c = 43$, and $m = 100$.

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- Try <https://xiaoweiz.shinyapps.io/randNumGen> for different parameters.



- An actual use of LCG ([Lewis et al. 1969](#)): $a = 7^5$, $c = 0$, $m = 2^{31} - 1 = 2,147,483,647$ (a prime number).
 - It adopts $u_i = \frac{x_i}{m+1}$.
 - It passes many of the standard statistical tests.
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- Note: By letting modulus m be a power of 2 (or close), the modulo operation can be conducted efficiently, since most digital computers use a binary representation of numbers.
- As computing power has increased, LCG is not adequate nowadays; more sophisticated RNGs are used in practice.

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- For 32-bit computers, [L'Ecuyer \(1988\)](#) suggests combining $J = 2$ generators with $a_1 = 40,014$, $m_1 = 2,147,483,563$, $a_2 = 40,692$, and $m_2 = 2,147,483,399$.

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 - 1 Select seed $x_{1,0}$ in the range $[1, m_1 - 1]$ for the first generator, and seed $x_{2,0}$ in the range $[1, m_2 - 1]$ for the second. Set $j = 0$.
 - 2 Calculate

$$x_{1,j+1} = a_1 x_{1,j} \bmod m_1,$$

$$x_{2,j+1} = a_2 x_{2,j} \bmod m_2.$$
 - 3 Let $x_{j+1} = (x_{1,j+1} - x_{2,j+1}) \bmod (m_1 - 1)$.
(Remark: mod uses floored division, i.e., $y \bmod m = y - m \lfloor \frac{y}{m} \rfloor$.)
 - 4 Return

$$u_{j+1} = \begin{cases} \frac{x_{j+1}}{m_1}, & \text{if } x_{j+1} > 0, \\ \frac{m_1 - 1}{m_1}, & \text{if } x_{j+1} = 0. \end{cases}$$
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It has cycle length $(m_1 - 1)(m_2 - 1)/2 \approx 2 \times 10^{18}$.



- Multiple Recursive Generator (MRG): Extends LCG by using a higher-order recursion:

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- A specific instance that has been widely implemented is MRG32k3a[†] ([L'Ecuyer 1999](#)), which is a *combined MRG* with $J = 2$ and $K = 3$.

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- A specific instance that has been widely implemented is MRG32k3a[†] ([L'Ecuyer 1999](#)), which is a *combined MRG* with $J = 2$ and $K = 3$.
 - It has cycle length $\approx 3 \times 10^{57}$, which is enormous.
 - If you could generate one billion (10^9) pseudo-random numbers per second, then it would take longer than the age of the universe to exhaust the period of MRG32k3a!

[†]MRG32k3a or its adaptation is one of the RNGs used in MATLAB, R, SAS, Arena, etc.



- Tests based on generated sequences of numbers.
 - *Frequency Test* for uniformity (discussed in next lecture)
 - Kolmogorov–Smirnov test (柯尔莫哥洛夫–斯米尔诺夫检验)
 - chi-square test (χ^2 test, 卡方检验)
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- Fortunately, the well-known RNGs which are widely used in simulation softwares and languages have been extensively tested and validated.
- Be careful when the RNG at hand is not explicitly known or documented!
 - Even RNGs that have been used for years in popular commercial softwares (e.g., Excel, Visual Basic), have been found to be inadequate ([L'Ecuyer 2001](#)).



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- Assumption: RNG is available, i.e. we have a sequence of random numbers (i.e., $\text{Unif}(0, 1)$ random variates).
- Goal: Produce random variates from a given probability distribution (e.g. exponential, Poisson, etc.).

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- Goal: Produce random variates from a given probability distribution (e.g. exponential, Poisson, etc.).
- Widely-used techniques[†]
 - Inverse-transform technique (generic)
 - Acceptance-rejection technique (generic)
 - Other ad-hoc methods for some specific distributions

[†] Methods introduced in this lecture are exact; there are also approximation methods such as MCMC.

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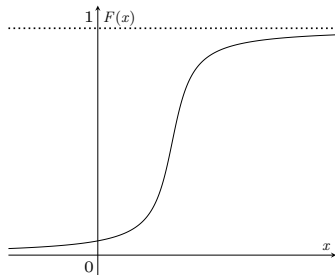


Figure: Continuous Random Variable

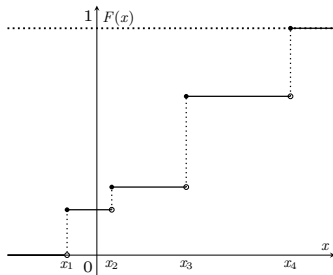


Figure: Discrete Random Variable

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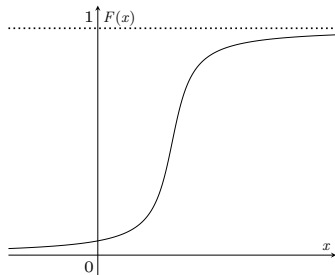


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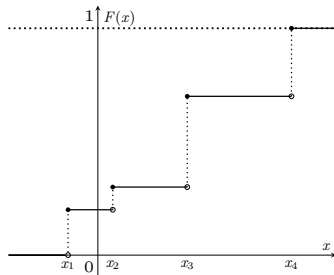


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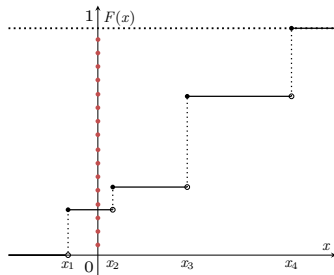
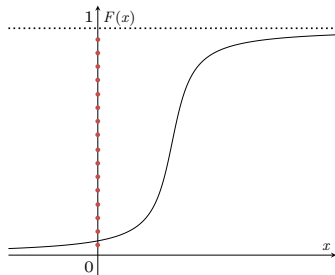


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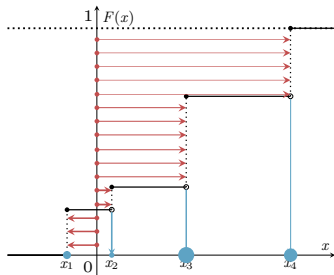
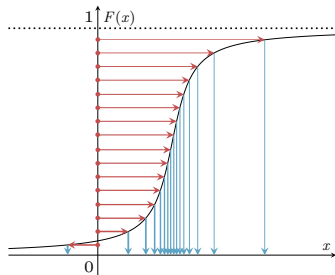


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 - 1 Generate (as needed) random numbers (on vertical axis).
 - 2 Map inversely to points on horizontal axis, which are the desired random variates from $F(x)$.

- The formal definition of inverse function is

$$F^{-1}(y) := \min\{x : F(x) \geq y\}, \quad 0 < y < 1.$$

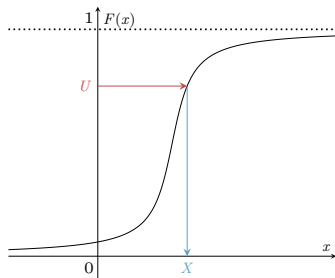


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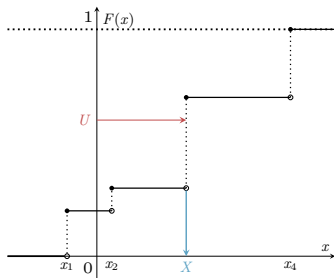


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$$\mathbb{P}(F^{-1}(U) \leq x) = \mathbb{P}(U \leq F(x)) = F(x).$$

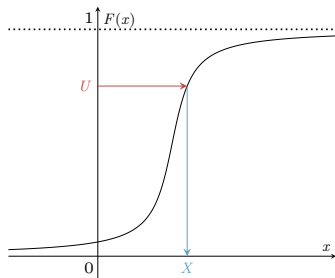


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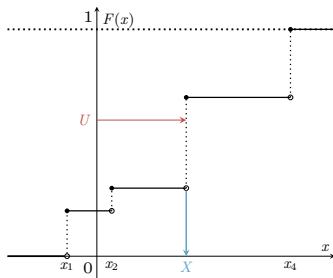


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- *Remark:* $1 - U \sim \text{Unif}(0, 1) \implies -\frac{1}{\lambda} \ln(U)$ is sufficient.
- Numerical test for $\text{Exp}(1)$ in **Excel**.
 - 1 Generate 200 random numbers.
 - 2 Obtain 200 random variates via the inverse function.



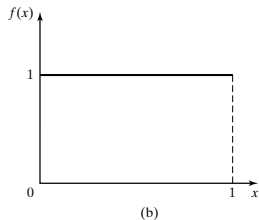
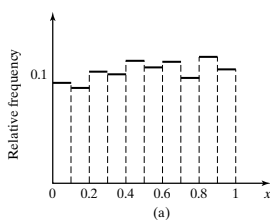


Figure:

- (a) Empirical histogram of 200 generated uniform random numbers;
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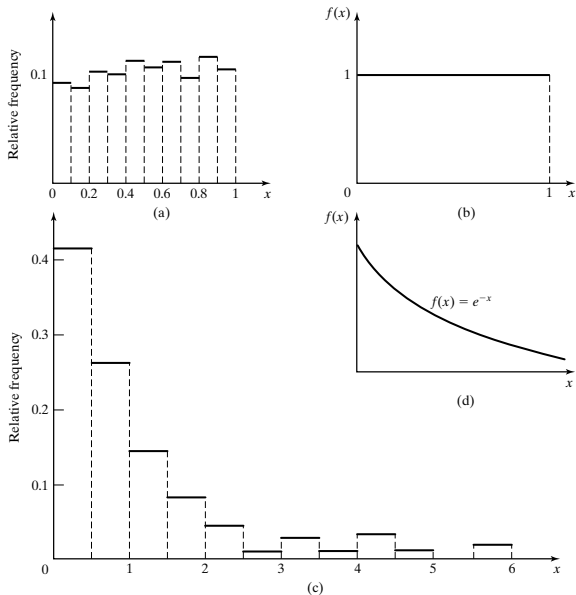


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- (a) Empirical histogram of 200 generated uniform random numbers;
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- (c) Empirical histogram of 200 generated variates from $\text{Exp}(1)$;
- (d) Theoretical density of $\text{Exp}(1)$.

(from [Banks et al. \(2010\)](#))



- Consider a discrete random variable X taking values 0, 1, 2 with probability 0.5, 0.3 and 0.2.

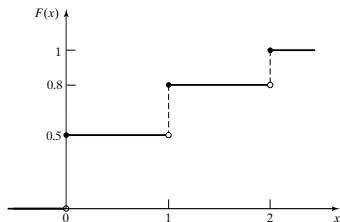
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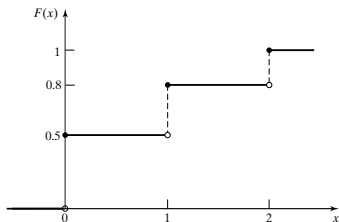
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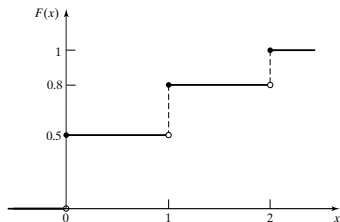


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Try it in Excel.



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 - Although you can solve the inverse transform via numerical methods anyway, the efficiency may be low.
- Acceptance-rejection technique is also useful for generating a *non-stationary Poisson process* (more details later).

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 - Whereas there exists a one-to-one mapping for the inverse-transform method.

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- For $1/4 \leq x \leq 1$,

$$\mathbb{P}\{U \leq x | U \geq 1/4\} = \frac{\mathbb{P}\{U \leq x \text{ and } U \geq 1/4\}}{\mathbb{P}\{U \geq 1/4\}} = \frac{x - 1/4}{3/4},$$

which is exactly the desired CDF of $X \sim \text{Unif}(1/4, 1)$.

- Suppose we want to generate random variates from X , whose density $f(x)$ has support $[a, b]$ and is upper bounded by M .

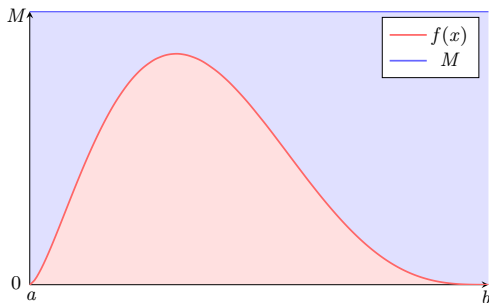


Figure: Bounded Support (original image from [ZHANG Xiaowei](#))

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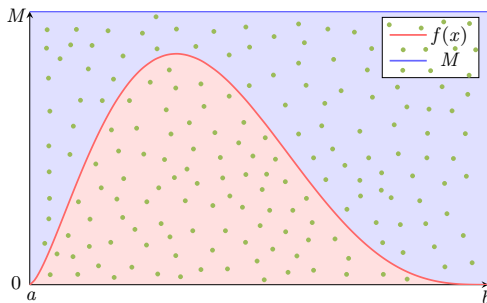


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- 1 Generate random variate pairs $(y_1, z_1), (y_2, z_2), \dots$, from $\text{uniform}\{(y, z) : a \leq y \leq b, 0 \leq z \leq M\}$.

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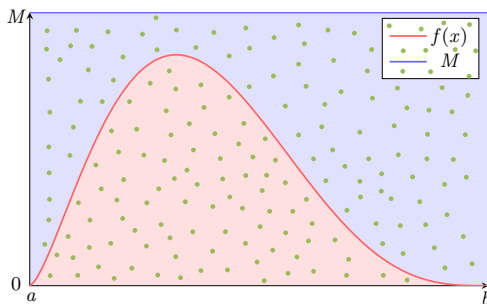


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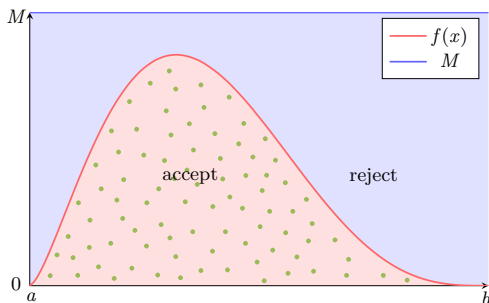


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- Accept the pair if $z_i < f(y_i)$ and output y_i as random variate from X with density $f(x)$.

- Y conditioned on the event $\{Z < f(Y)\}$ has the same distribution as X , i.e., having density $f(x)$.
 - $(Y, Z) \sim \text{uniform}\{(y, z) : a \leq y \leq b, 0 \leq z \leq M\}$.

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$$\mathbb{P}\{Y \leq x | Z < f(Y)\}$$

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 &= \frac{\int_a^x \int_0^{f(y)} f_{Y,Z}(y, z) dz dy}{\int_a^b \int_0^{f(y)} f_{Y,Z}(y, z) dz dy} \quad \text{Note: } f_{Y,Z}(y, z) = \frac{1}{(b-a)M} \\
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Proof.

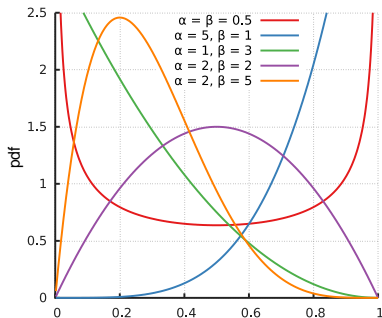
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- The acceptance rate is $\mathbb{P}\{Z < f(Y)\} = \frac{1}{(b-a)M}$.

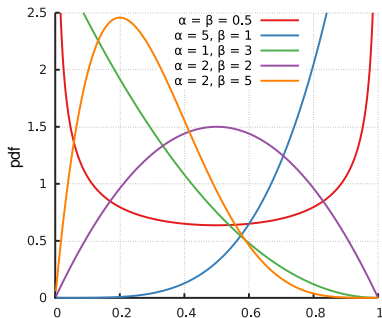


- Goal: Generate random variates from $\text{Beta}(\alpha, \beta)$, where the density is $f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)}$, $x \in [0, 1]$.

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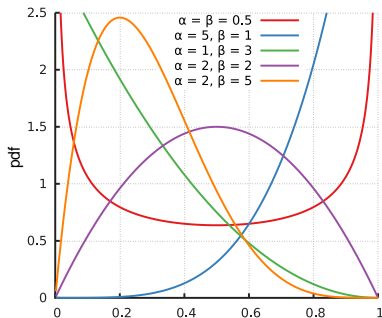


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- The acceptance rate is $\frac{1}{(b-a)M} = \frac{1}{(1-0)M} = \frac{1}{M}$.

- Generate random variates from X , whose density $f(x)$ is upper bounded by $Mg(x)$, where $g(x)$ is *instrumental* density.

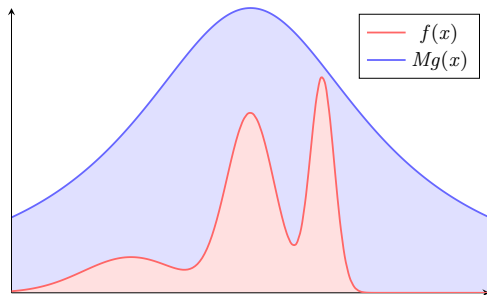


Figure: Unbounded Support (original image from [ZHANG Xiaowei](#))

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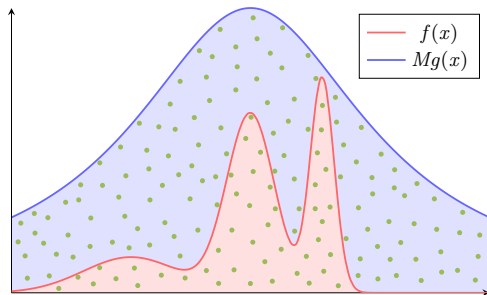


Figure: Unbounded Support (original image from [ZHANG Xiaowei](#))

- 1 Generate random variate pairs $(y_1, z_1), (y_2, z_2), \dots$, from $\text{uniform}\{(y, z) : y \in \text{support of } g(\cdot), 0 \leq z \leq Mg(y)\}$.

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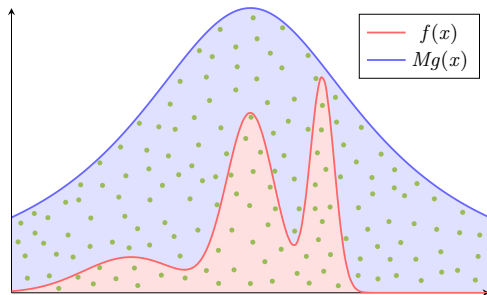


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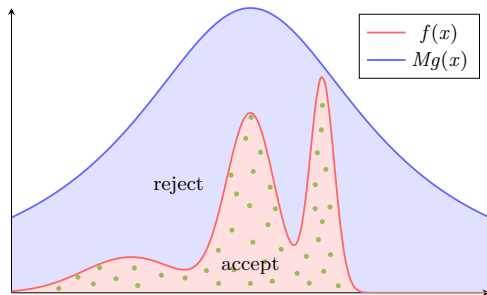


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- Accept the pair if $z_i < f(y_i)$ and output y_i as random variate from X with density $f(x)$.

- Y conditioned on the event $\{Z < f(Y)\}$ has the same distribution as X , i.e., having density $f(x)$.
 - Let Θ denote $\{(y, z) : y \in \text{support of } g(\cdot), 0 \leq z \leq Mg(y)\}$.
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- The acceptance rate is

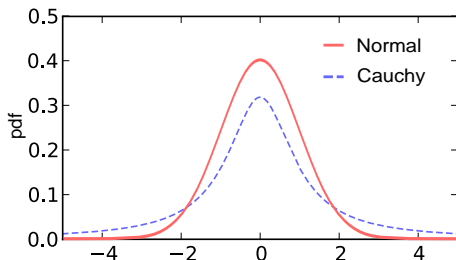
$$\mathbb{P}\{Z < f(Y)\} = \frac{1}{\Theta \text{ area}} = \frac{1}{\int_{-\infty}^{\infty} Mg(y) dy} = \frac{1}{M \int_{-\infty}^{\infty} g(y) dy} = \frac{1}{M}$$



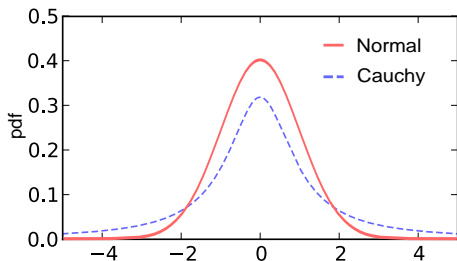
- Goal: Generate random variates from $\mathcal{N}(0, 1)$, where the density is $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, $x \in (-\infty, \infty)$.

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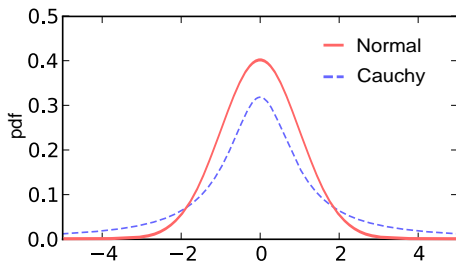


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- It is easy to see that $\frac{f(x)}{g(x)} = \sqrt{\frac{\pi}{2}}(1+x^2)e^{-\frac{x^2}{2}}$ is maximized at $x = \pm 1$ and the maximum is $\sqrt{\frac{2\pi}{e}}$, which is the required M .

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- The acceptance rate is $\frac{1}{M} = \sqrt{\frac{e}{2\pi}} \approx 0.6577$.

- Box–Muller method for $\mathcal{N}(0, 1)$ random variates:
 - ① Generate u_1 and u_2 independently from $\text{Unif}(0, 1)$.
 - ② Let $z_1 = \sqrt{-2 \ln u_1} \cos(2\pi u_2)$ and $z_2 = \sqrt{-2 \ln u_1} \sin(2\pi u_2)$.

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- Intuition:
 - For two independent $\mathcal{N}(0, 1)$ RVs Z_1 and Z_2 ,
$$Z_1^2, Z_2^2 \sim \chi_1^2, \quad Z_1^2 + Z_2^2 \sim \chi_2^2.$$
 - $X \sim \text{Exp}(1/2) \iff X \sim \chi_2^2$.
 - $-2 \ln u_1$ is a random variate from $\text{Exp}(1/2)$ (and thus χ_2^2).
 - The angle is distributed uniformly around the circle.

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- z_1 and z_2 are random variates from $\mathcal{N}(0, 1)$ (independent).
- Intuition:
 - For two independent $\mathcal{N}(0, 1)$ RVs Z_1 and Z_2 ,
$$Z_1^2, Z_2^2 \sim \chi_1^2, Z_1^2 + Z_2^2 \sim \chi_2^2.$$
 - $X \sim \text{Exp}(1/2) \iff X \sim \chi_2^2$.
 - $-2 \ln u_1$ is a random variate from $\text{Exp}(1/2)$ (and thus χ_2^2).
 - The angle is distributed uniformly around the circle.
- Rigorous proof.

- Box–Muller method for $\mathcal{N}(0, 1)$ random variates:
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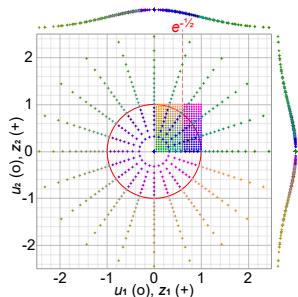


Figure: Box–Muller Method Visualisation
(image by Cmglee / CC BY 3.0)

[Interactive Graph](#)

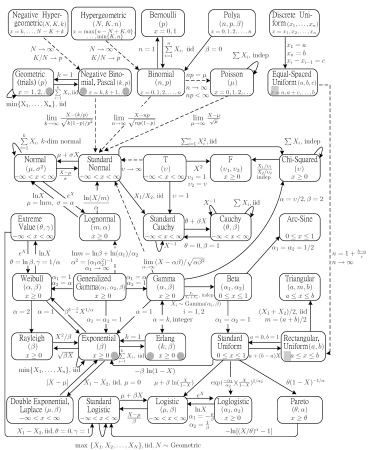


Figure: Relationships Among 35 Distributions (from Song (2005))

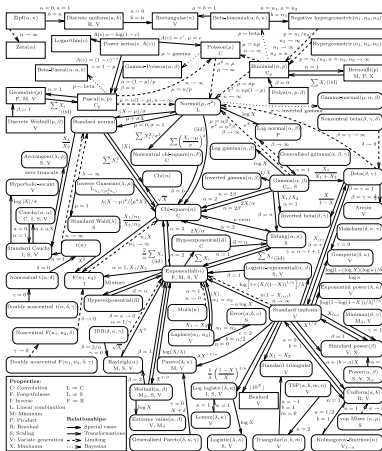


Figure: Relationships Among 76 Distributions (from Leemis & McQueston (2008))



- **Poisson process** with rate λ : Interarrival time distribution is exponential with rate λ (or mean $1/\lambda$), and

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- To generate Poisson process with rate λ , one only need to generate iid $\text{Exp}(\lambda)$ random variates.
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- **Nonhomogeneous Poisson process** with rate (intensity) function $\lambda(t)$:

$$N(t+h) - N(t) \sim \text{Poisson}(m(t+h) - m(t)),$$

where $m(t) = \int_0^t \lambda(s) ds$.

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If $u \leq \lambda(t)/\lambda^*$, then $s_i = t$ and $i \leftarrow i + 1$.
 - 4 Go to Step 2.